Discrete Mathematics 93 (1991) 183–194
North-Holland

A-cordial graphs

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Received 29 March 1989

Abstract

Hovey, M., A-cordial graphs, Discrete Mathematics 93 (1991) 183–194.

We introduce A-cordial graphs, for an abelian group A. (If $A = \mathbf{Z}_k$ we call them k-cordial graphs.) These generalize harmonious, elegant, and cordial graphs. They also provide a graph-theoretic realization of the function v_γ studied by Graham and Sloane (1980). We show that trees are 3, 4 and 5-cordial and provide a finite (though long) test that, if passed, guarantees that all trees are A-cordial. We conjecture that trees are k-cordial for all k. We provide a partial classification of which cycles and complete graphs are k-cordial, and we show that for k even and >4, most graphs are not k-cordial.

1. Introduction

Graph labellings are the source of many easily stated yet unsolved problems. They have a very extensive literature. The paper of Graham and Sloane [5] has a good bibliography of the literature published before 1980.

All our graphs will be finite and simple; that is, without multiple edges or loops. G will always denote a graph, T a tree, V the vertex set, and E the edge set. We will always use n, e for the number of vertices and edges respectively. Let A be an abelian group. A *vertex-labelling* of G (with elements from A) is simply a function $f: V \to A$. Any vertex-labelling induces an edge-labelling which we also denote by f, defined by $f(v, w) = f(v) + f(w)$ for an edge (v, w) of G. The vertex-labelling together with its induced edge-labelling we will call simply a labelling of G.

Obviously there are many labellings of G. The question usually is: Are there evenly packed labellings? To make this precise, we need some notation. For $a \in A$ let v_a, e_a denote the number of vertices and edges, respectively, which are labelled a.

Definition 1. G is A-cordial if there is a labelling $f: V \to A$ such that for all $a, b \in A$ we have:

(1) $|v_a - v_b| \leq 1$; and

(2) $|e_a - e_b| \leq 1$.

If $A = \mathbf{Z}_k$ we will say that G is k-cordial.

Then one wants to know which graphs are A-cordial. Special cases of this have appeared in the literature before. Recall that n, e denote the number of vertices and edges of G.

(A) G is harmonious [5] if and only if G is e-cordial.

(B) G is elegant [2] if and only if G is n-cordial.

(C) G is cordial [1] if and only if G is 2-cordial.

The problem of completely classifying A-cordial graphs seems to be a very hard one. For example, Graham and Sloane, who introduced harmonious graphs in [5], find many examples of harmonious graphs and many examples of graphs which are not harmonious. The best general result they get is that most graphs are not harmonious, and they conjecture that all trees are harmonious. Chang, Hsu, and Rogers, who introduced elegant graphs in [2], also conjecture that all trees are elegant. Cahit introduced cordial graphs in [1] in part because harmonious graphs seemed rather intractable, and he proves that trees are cordial. We can unify these conjectures with the following conjecture.

Conjecture 1. Trees are k-cordial for all k.

We point out that this conjecture can be false for more general values of A. Indeed, the 4-vertex path is not $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-cordial. Thus, we will usually restrict attention to $A = \mathbf{Z}_k$, though those results which hold for more general A are noted.

In the next section of this paper we deal with some preliminaries. For $k \neq 3$ we find a graph which is not k-cordial. We relate k-cordial graphs to the function v_γ considered by Graham and Sloane [5]. This function will be defined in the next section, but informally it deals with packing $\mathbf{Z}_k$ with sums. Section 3 deals with trees. We show that caterpillars are k-cordial for all k. We show that there is a finite test which can show that all trees are k-cordial, and we use variants of it to show all trees are 3, 4, and 5-cordial. Section 4 deals with cycles, and the last with complete graphs. The last section also contains a construction for making new k-cordial graphs from old, and also in it we show that for k even and >4 most graphs are not k-cordial.

2. Preliminaries

First note the following two lemmas, which are slight generalizations of Theorem 8 of [5], and whose proofs are the same.

Lemma 1. *If f is an A-cordial labelling of G so is $f + a$ for any $a \in A$.*

Lemma 2. *If A is a ring and f is an A-cordial labelling of G so is af for any unit $a \in A$.*

Recall the function $v_\gamma(n)$ considered by Graham and Sloane.

Definition 2. $v_\gamma(n)$ is the smallest k such that there is an n-element subset $B=\{0=b_1<\cdots<b_n\}$ of $\mathbf{Z}_k$ such that any $r\in\mathbf{Z}_k$ can be written as $r=b_i+b_j$, $i<j$ in at most one way.

A simple comparison of this with the definition of k-cordial gives the following.

Theorem 1. *$v_\gamma(n)$ is the smallest $k\geqslant n$ such that all graphs on n vertices are k-cordial.*

Combining this with the bounds on v_γ given by Graham and Sloane we have the following.

Corollary 1. *A graph on n vertices is k-cordial for all $k>n^2+\mathrm{O}(n^{36/23})$.*

If the graph happens to be harmonious we can reduce this bound while also removing its asymptotic nature. Recall that e denoted the number of edges.

Proposition 1. *If G is harmonious then G is k-cordial for $k\geqslant 2e$.*

Proof. Consider a harmonious labelling of G. If G is a tree, change the labelling so that 0 is the repeated vertex label, then replace one of the zeroes with e. The edge-labels are all distinct $\bmod e$, so they are distinct integers all $\leqslant 2e-1$. Thus they are distinct $\bmod k$ for all $k\geqslant 2e$. $\square$

Remark 1. For $k\neq 3$ there are graphs which are not k-cordial.

Indeed, consider the complete bipartite graph $K_{2,k-2}$. Call the two vertices in one part v, w. Suppose we have a k-cordial labelling. We can assume v is labelled 0. Suppose w is labelled a. Then there is no way to get a as an edge-label so we must have $2(k-2)\leqslant k$. That is, we must have $k\leqslant 3$. It is easy to find graphs which are not 2-cordial; a 6-cycle will do. However, I have found no graph which is not 3-cordial.

Conjecture 2. All connected graphs are 3-cordial.

We provide some favorable evidence for this conjecture in later sections.

3. Trees

Our first result is an extension of a theorem of Graham and Sloane. Recall that a *caterpillar* is a tree with a central path such that every vertex is at distance at most one from the central path.

Theorem 2. *Caterpillars are k-cordial for all k.*

Proof. We use the sequential labelling due to Grace [4]. Recall that this is obtained by drawing the caterpillar as a planar bipartite graph with say m vertices on the left, and the central path starting on the left. Then label the left hand side $1, \ldots, m$, starting at the top, and label the right hand side $m+1, \ldots, n$ again starting at the top. The edge-labels are then $m+2, \ldots, m+n$. Now we simply reduce mod k to get a k-cordial labelling. □

The simplest method to prove that all trees are k-cordial would be induction. It seems natural to try to add k vertices at a time, trying to label each vertex differently. This leads one to consider subforests of the tree.

Definition 3. A *rooted forest* F is a finite collection of disjoint rooted trees (T_i, r_i). We do not consider the roots to be vertices of the forest. This means that a rooted forest has the same number of vertices as edges.

We now define what it means for a rooted forest to be A-cordial.

Definition 4. A rooted forest F with vertex set V, edge set E, and root set R is said to be A-cordial if for every $g: R \to A$ and every $l \in A$ there is an $f: V \to A$ satisfying the following conditions:

(1) $|v_i - v_j| \leqslant 1$ for all $i, j \in A$;

(2) $|e_i - e_j| \leqslant 1$ for all i, j with neither equal to l; and

(3) $0 \leqslant e_l - e_i \leqslant 2$ for all i.

Again, if $A = \mathbf{Z}_k$ we will say that F is a k-cordial.

Essentially, F is A-cordial if and only if no matter how one defines the labelling on the roots, there is an A-cordial labelling on the rest of the forest. At first glance it seems as if the third condition in the definition should not be there: that is one should just require that $|e_i - e_j| \leqslant 1$ for all i, j. However, as with trees, we would like all rooted forests to be k-cordial. There are many rooted forests which would not be k-cordial under the stronger definition. For example, the path on five vertices with the root being the middle vertex would not be 4-cordial under the stronger definition. Any vertex labelling using all four labels must repeat an edge-label.

The importance of these rooted forests is that they are precisely what is needed to extend a k-cordial labelling of a small tree to a bigger one containing the small one.

Theorem 3. *Suppose $|A| = k$. If all trees and rooted forests with k vertices are A-cordial, then all trees are A-cordial.*

Proof. First notice that if all trees on k vertices are A-cordial, so are all trees with less then k vertices. Indeed, one only has to restrict the labelling to subtrees. Since each vertex and edge label appears at most once, the same will be true in the subtree. A similar remark holds for the rooted forests.

Now we proceed by induction on m. Suppose all trees on mk vertices ($m \geq 1$) are A-cordial. Suppose we have a tree T on n vertices with $mk < n \leq (m+1)k$. We wish to find an A-cordial labelling of T. We can obtain T from a subtree T' on mk vertices by attaching a forest F with $\leq k$ vertices. Let the roots of F be the vertices it is attached to in T'. By the inductive hypothesis, there is an A-cordial labelling of T'. This induces a labelling on the roots of F. Let l be the edge-label that appears least often in T'. Then since F is A-cordial, we can extend the labelling of the roots to an A-cordial labelling of F with l possibly occurring twice as an edge-label, but every other vertex label and edge label appearing at most once. This labelling, together with the A-cordial labelling of T', defines an A-cordial labelling of T. □

Note that k-cordiality of a graph with k vertices is equivalent to the elegance of that graph. If we extend the definition of elegance to rooted forests as above, we can rephrase this theorem.

Theorem 4. *If all trees and rooted forests are elegant, then all trees are k-cordial for all k.*

This provides a finite way of checking if all trees are k-cordial. Of course, if it fails one cannot be sure that in fact there is a tree that is not k-cordial. However in practice this does not arise.

Conjecture 3. All trees and rooted forests are elegant.

In the cases that follow we use the idea behind this theorem to show that all trees are 3, 4, and 5-cordial. At least in these cases we can cut down the number of rooted forests we have to check considerably.

First we prove an easy lemma.

Lemma 3. *If all trees on mk vertices are k-cordial then all trees T with $mk \leq |T| \leq mk + [k/2] + 1$ are k-cordial.*

Proof. If we attach a leaf to a tree with $mk + j$ vertices we have $k - j$ choices for the vertex label that will preserve vertex-k-cordiality. In order to preserve edge-k-cordiality we must avoid $j - 1$ edge labels if $j > 0$. We can do this as long as $k-j>j-1$. If $j=0$ we have only one choice for the edge label but there are no restrictions on the vertex label. □

Letting $m=0$ in this lemma we obtain the following corollary, which provides some support for the harmonious tree conjecture in light of Proposition 1 of the last section.

Corollary 2. *If the tree T has n vertices, it is k-cordial for $k \geqslant 2(n-1)$.*

Note that Lemma 3 implies that all trees are 2-cordial, recovering the result of Cahit in [1]. We can also use this lemma to show that all trees are 3, 4, and 5-cordial.

Theorem 5. *All trees are 3-cordial.*

Proof. Every tree on 3 vertices is a caterpillar, so is 3-cordial. By the lemma it is enough to show that if all trees on $3m$ vertices are 3-cordial, then all trees on $3m+3$ vertices are 3 cordial. Let T be a tree with $3m+3$ vertices. If T is a path, then it is a caterpillar and so is 3-cordial. So we can assume T has 3 leaves, say l_0, l_1, l_2 connected to v_0, v_1, v_2 respectively. Delete them and label the resulting tree 3-cordially. Let the labels on the v_i be denoted a_i. Then we can assume, by permuting the v_i and by Lemmas 1 and 2, that $(a_0, a_1, a_2) = (0,0,0)$, $(0,0,1)$ or $(0,1,2)$. Suppose that edge-label j appears $m-1$ times while the other two edge-labels appear m times. We must find a way of labelling l_0, l_1, l_2 with distinct elements so that j appears as an edge-label and no other edge-label appears twice, though j itself might. We do this case by case. Each case is presented as an array with the top row being the a_i, the middle row the labels on the l_i, and the bottom row the induced edge-labels. These cases correspond to showing that rooted forests on 3 vertices in which each tree is a star are 3-cordial.

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array} \qquad \begin{array}{ccc} 0 & 0 & 1 \\ j & j+1 & j+2 \\ j & j+1 & j \end{array} \qquad \begin{array}{ccc} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{array} \quad \square$$

Theorem 6. *All trees are 4-cordial.*

Proof. We proceed in the same manner. Every tree on $\leqslant 4$ vertices is a caterpillar, so is 4-cordial. By the lemma, we only need show that trees with $4m$ vertices are 4-cordial implies trees with $4m+4$ vertices are 4-cordial. Let T have $4m+4$ vertices. We will first deal with the case where T has four leaves. We will present these cases as above.

0	0	0	0	0	0	0	1
0	1	2	3	j	$j+1$	$j+2$	$j+3$
0	1	2	3	j	$j+1$	$j+2$	j

0	0	0	2	0	0	1	1
j	$j+1$	$j+3$	$j+2$	j	$j+1$	$j+2$	$j+3$
j	$j+1$	$j+3$	j	j	$j+1$	$j+3$	j

0	0	2	2	0	0	1	2
0	2	1	3	j	$j+3$	$j+1$	$j+2$
0	2	3	1	j	$j+3$	$j+2$	j

0	0	1	3	0	1	2	3
0	1	2	3	j	$j+3$	$j+1$	$j+2$
0	1	3	2	j	j	$j+3$	$j+1$

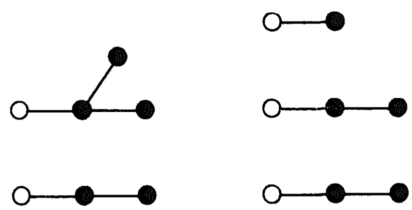
If T does not have four leaves we assume it has three, since otherwise it would be a caterpillar. Let l, l' be the end-vertices of a maximal length path in T. Then l, l' are leaves. Let them be connected to v, v'. If there is any connected to v not on the maximal path, it must be an end-edge. But there are only 3 leaves so in this case T is a caterpillar. Thus we can assume $\deg(v) = \deg(v') = 2$. (And $v \neq v'$, for if they are equal, T is a caterpillar.) Let w, w' be the other vertex connected to v, v'. Remove l, l', v, v' and label the resulting tree 4-cordially. We can assume, by changing the labelling, that the labels of w, w' are one of $(0,0)$, $(0,1)$, $(0,2)$. We must find an extension of this labelling. Recall that j is the missing edge label. If the labels of w, w' are $(0,0)$ or $(0,2)$ we can assume $j \neq 3$, but we have to do each of the remaining cases separately. There are seven cases total, three each for $(0,0)$, $(0,2)$ and one for $(0,1)$ where we can find an extension which uses each edge label once. The case when $j = 0$ and w, w' (the empty circles) are both labelled 0 is shown below. The others are similar.

This completes the proof. □

Theorem 7. *All trees are 5-cordial.*

Proof. This time the lemma only allows us to go from $5m$ vertices to $5m+3$ vertices. So we have to consider both $5m+4$ and $5m+5$ vertices. The first step is to show we can attach 4 or 5 leaves to a $5m$ vertex tree and preserve 5-cordiality. This is exactly analogous to the previous theorems and is easily checked, though there are more cases.

We then proceed as in the 4-cordial case to deduce that if a tree on $5m+5$ vertices with less than 5 leaves is not a caterpillar it must contain one of the two following subforests. Here an empty circle indicates that the vertex may have other edges adjacent to it in the tree. It is not necessary that these vertices be distinct.



Now we have to check that given any labelling of the roots of these subforests (the empty circles), and any value of j (the least frequent edge label), there is a labelling of the remainder of the subforest using each vertex label once, using j as an edge-label, and not using any other edge-label twice. This is easily, though laboriously, checked. This will also take care of the $5m+4$ vertex case, as any such tree with less than 4 leaves that is not a caterpillar must contain a subforest of one of the two above forests. □

4. Cycles

Theorem 8. *Odd cycles with leaves attached are k-cordial for all k.*

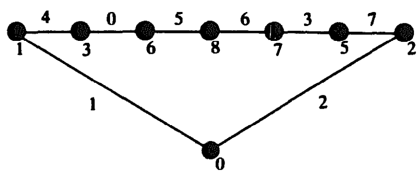
Proof. This is very similar to the analogous result for caterpillars. An odd cycle with leaves attached is simply a caterpillar with an extra edge. The sequential labelling given for caterpillars extends to a sequential labelling here, as pointed out by Grace [4]. Then we reduce mod k. □

Theorem 9. *Cycles are k-cordial for all odd k.*

Proof. Consider the $mk+j$ cycle, where $0 \leqslant j < k$. First we assume $m \geqslant 1$. In this case we label the cycle as follows: Break the cycle into j blocks of $m+1$ vertices and $k-j$ blocks of m vertices. Label every vertex in the ith block $i-1$. This evenly splits the vertex labels. Any edge label of the form $2i$ for $i \leqslant j$ appears $m+1$ times: n times in the block labelled i and once on the border between block $(2i-1)/2$ and block $(2i+1)/2$. Similarly, any edge label of the form $2i$ for $i > j$ appears m times. Since 2 is a unit, this takes care of every label.

Now suppose $m=0$, so we are considering the j-cycle for $j<k$. By the preceding theorem we can assume j is even. This time we will divide the cycle into 4 blocks. If j is divisible by 4 they will each have size $j/4$; if not, the first and third will have size $(j-2)/4$ while the second and fourth will have size $(j+2)/4$. Label

the ith element of the first block $2i-1$. Label the ith element of the second block $k-1+2(i-l)$ where l is the number of elements in the block. Label the ith element of the third block $k-2i$. Finally, label the ith element of the fourth block $2l-2(i-1)$ where this is the same l. The first block contains only odd numbers less than $k/2$, while the third block contains only odd numbers greater than $k/2$. The second block contains only even numbers greater than $k/2$, while the fourth block contains only even numbers less than $k/2$. From this the k-cordiality of the labelling follows. Perhaps an example is in order, say with $j=8$ and $k=9$.



This completes the proof. □

Theorem 10. *Suppose k is even. Then the $2mk+j$ cycle, where $0 \leq j < 2k$, is k-cordial for $j \leq k/2+2$ and for $j > k$. It is not k-cordial for $j=k$.*

Proof. First we show that it is not k-cordial for $j=k$. For any labelling f, we have

$$\sum_{e \in E} f(e) = \sum_{v \in V} d_v f(v)$$

where d_v is the degree of v. Applying this to a k-cordial labelling of the $2mk+k$ cycle we find

$$(2m+1)k(k-1)/2 \equiv 2(2m+1)k(k-1)/2 \bmod k.$$

The right-hand side of this is 0 while the left-hand side is not, so there is no such labelling.

Before continuing with the proof, we prove a lemma.

Lemma 4. *Suppose k even. Then if the j cycle is k-cordial, so is the $2mk+j$ cycle for all m.*

Proof. For each block of size $2k$ we use the following labelling. Divide the block into k subblocks of size 2. For $i \leq k/2$ label the ith subblock with $i-1$, $i-1+k/2$. This uses each vertex-label once and is sequential on the edge-labels, missing only $k/2-1$. For $i > k/2$ label the ith subblock like the $i-k/2$th block, only reversed. These blocks use each vertex-label equally often and fit together to use each edge-label equally often except $k/2-1$, which is the last vertex label. Now if we have a k-cordial labelling of the j cycle, we can assume 0 appears as a vertex label. Simply label the remainder of the $2mk+j$ cycle with the labels from the j cycle, starting with 0. This is a k-cordial labelling of the $2mk+j$ cycle. □

We return to the proof of the theorem now. We can assume j even. If $j > k$, we can use the pattern in the above theorem to label the j cycle. The pattern will be incomplete, but every vertex label will appear once or twice, and every edge-label will appear once or twice as well.

If $j \leq k/2+2$ we label the j cycle with $0, 1, \ldots, j-2, j$. The edge labels will then be the odd integers $\leq 2j-5$ and $2j-2, j$ which are even. The bound on j ensures that none of the odd integers are repeated. □

Conjecture 4. For k even the $2mk+j$ cycle, where $0 \leq j < 2k$, is k-cordial if and only if $j \neq k$.

5. Other graphs

We begin with a method for constructing new A-cordial graphs from old ones. This method is a generalization of a construction of Lee and Liu [3] who were interested in cordial graphs.

Proposition 2. *Suppose $|A| = k$, and G, H are A-cordial, with k dividing the number of edges of H. Fix a particular A-cordial labelling of H. Suppose in addition that H can be divided into m disjoint subgraphs H_i each of which is vertex-A-cordial under the given labelling and has k dividing $|H_i|$. (That is each vertex-label appears equally often.) Then for any division of G into m subgraphs G_i, not necessarily disjoint, the graph obtained by joining G_i to H_i for all i is A-cordial.*

Proof. The proposition is arranged precisely so that the old A-cordial labellings of G and H give an A-cordial labelling of the new graph. □

We can now use this to investigate the k-cordiality of complete graphs.

Theorem 11. *If k is odd, K_{mk+j}, where $0 \leq j < k$, is k-cordial if and only if K_j is k-cordial. In particular, all complete graphs are 3-cordial.*

Proof. First notice that K_k is k-cordial. Indeed, there is only one possibility for the vertex labelling, and, since k is odd, each edge-label will appear $(k-1)/2$ times. Now the preceding proposition implies that K_j being k-cordial forces K_{mk+j} to be k-cordial.

For the converse, suppose K_{mk+j} is k-cordial. Let $S = \{i \mid v_i = m+1\}$. So $|S| = j$. For each $i \in S$ choose a vertex w_i labelled i, and let $W = \{w_i\}$. If we remove W we have the (unique) k-cordial labelling of K_{mk}. Thus every edge-label appears equally often. If we look at the edges involving exactly one element of

W, we find that each edge-label appears equally often there as well. Thus the subgraph induced by W is k-cordial. This subgraph is K_j. □

Note that there are many times when K_j is not k-cordial. In fact Graham and Sloane show that K_j is not harmonious for $j \geq 5$. If k is even something very different happens.

Theorem 12. *If $k > 4$ is even, most graphs are not k-cordial.*

Proof. Suppose our graph has $mk + j$ vertices, where $j < k$. What is the maximum number of even edge-labels that could occur? There are $k/2 - 1$ 2-subsets of $\mathbf{Z}_k$ whose sum is a particular even element. Computation yields that the total number of even edges is $\leq$

$$\frac{k}{2}m^2 - m + jm + \frac{k(k-2)}{4} = O\left(\frac{k}{2}m^2\right)$$

Thus the total number of edges is a k-cordial graph on $mk + j$ vertices is $\leq O(km)^2$. Since the total number of edges is $O(\frac{1}{2}k^2m^2)$, graphs containing more than $2/k$ of the total number of vertices will not be k-cordial. □

Theorem 13. *If k is even, K_{mk} is k-cordial if and only if $m = 1$. Also K_{mk+j}, where $0 \leq j < k$, is not k-cordial for $m > (k+2)/4$.*

Proof. The only possible k-cordial labelling of K_{mk} is to have each vertex label appearing m times. The number of edges labelled with any even label, say 0, in this labelling, is

$$m(m-1) + (k/2 - 1)m^2 = (k/2)m^2 - m.$$

The number of edges labelled with any odd label, say 1, is $(k/2)m^2$. Thus K_{mk} is k-cordial if and only if $m = 1$.

The bound in the second part of the theorem is obtained as in the previous theorem. □

Note added in proof

Joe Gallian points out that the definition of elegant in [2] requires that the edge labels be nonzero. Thus one should read 'n-cordial' wherever 'elegant' appears in this paper.

Acknowledgment

The author would like to thank K. Collins for many helpful discussions.

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